Limit Laws for Random Spatial Graphical Models

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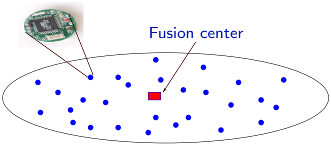
Introduction

Motivating Examples

- Sensor network collecting data.
- Social network in a physical geographic area.

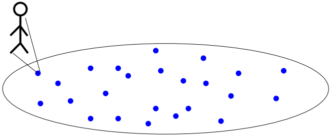


Locations of Nodes

- Irregular, far from a lattice
- Random node placement

Observations at Nodes

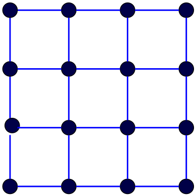
- Correlated observations
- Dependent on locations of nodes
- Assumed to be a graphical model



Motivating Example: Ising Model

Ising Model

- Used to model attractive forces between molecules
- Change of state related to phase transition



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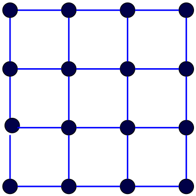

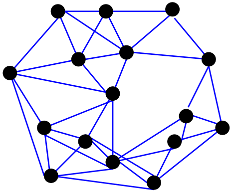

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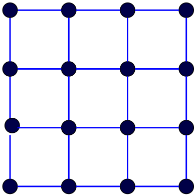
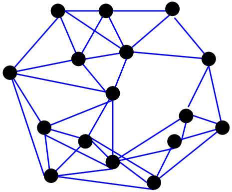


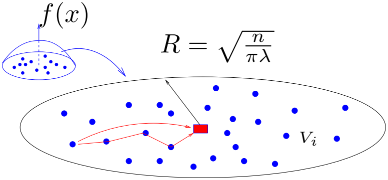


Graphical Models under Random Node Placement

- How does irregular and random node placement affect properties compared to lattice structure?
- How do the limits of functions of observations depend on node placement and when do they exist?

Brief Problem Description

Consider n randomly distributed nodes at locations $V_i \in \mathbf{V}_n$ making random observations $\mathbf{Y}_{\mathbf{V}_n}$.



Random Node Locations and Euclidean Graphs

- Points $X_i \overset{\text{i.i.d.}}{\sim} f(x)$ on unit ball $\mathcal{B}_1$
- Network scaled to a fixed density λ: $V_i = \sqrt{\frac{n}{\lambda}} X_i$
- $\mathcal{G}(\mathbf{V}_n)$ is a graph on $\mathbf{V}_n$

Brief Problem Description Contd.,

Correlated Observations as a Graphical Model

$\mathbf{Y}_{\mathbf{V}_n}$ is a graphical model (Markov random field) with $\mathcal{G}(\mathbf{V}_n)$ as the graph.

Functions of Node Observations

- Locally-defined functions involving sums of terms at individual nodes

  Example: mean value of the observations.

$$\frac{1}{n}\sum_{i=1}^{n}\xi((V_i, Y_i); (\mathbf{V}_n, \mathbf{Y}_n)), \quad n \to \infty.$$

- When do the limits exist?
- How do the limits depend on graph $\mathcal{G}$ and node placement distribution f?

Summary of Results

Limits of Functions of Random Spatial Graphical Models

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\mathbb{E}[\xi((V_i, Y_i); (\mathbf{V}_n, \mathbf{Y}_n))] = \xi_\infty.$$

Conditions for Existence of Limits

- ξ has bounded moments
- Weak dependence on data and position of nodes far away
- Related to degree-dependent percolation

Relationship between Limiting Behavior and Graph Randomness

- Limiting constant ξ_∞, as an explicit function of node placement distribution f and graph $\mathcal{G}$

Related Work

Classical Works on Gibbs Measures on Lattices

- Ising model is the most understood model
- Books by Georgii, Simon etc.,

Gibbs Measures on Trees

Phase transition threshold based on maximum degree (Weitz)

Recent Works on Gibbs Measures on Random Graphs

- Phase transitions in sparse random graphs (Dembo & Montanari, Mossel & Sly)
- Utilize locally tree-like property

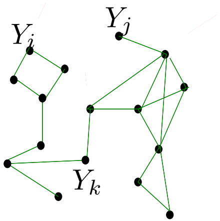
Limit Laws on Euclidean Random Graphs

- Graphs such as k-NNG, geometric random graph
- CLT, LLN for graph functionals (Penrose & Yukich)
- Do not cover correlated variables on random graphs

Outline

Dependency Graph and Graphical Model

- Consider an undirected graph $\mathcal{G}(\mathbf{V})$, each vertex $V_i \in \mathbf{V}$ is associated with a random variable Y_i



Graphical Models also known as Markov Random Fields.

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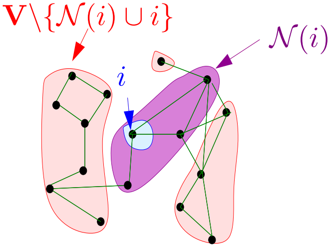
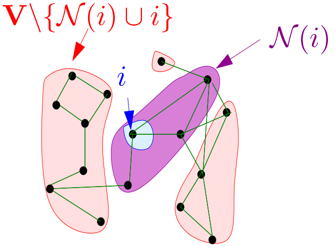


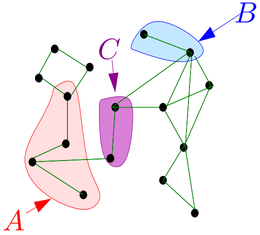
$$\mathbf{Y}_i \perp\!\!\!\perp \mathbf{Y}_{\mathbf{V}\setminus\{\mathcal{N}(i)\cup i\}} | \mathbf{Y}_{\mathcal{N}(i)}$$

Graphical Models also known as Markov Random Fields.

Dependency Graph and Graphical Model

- Consider an undirected graph $\mathcal{G}(\mathbf{V})$, each vertex $V_i \in \mathbf{V}$ is associated with a random variable Y_i
- For any disjoint sets A, B, C such that C separates A and B,





$$\mathbf{Y}_i \perp\!\!\!\perp \mathbf{Y}_{\mathbf{V}\backslash\{\mathcal{N}(i)\cup i\}} | \mathbf{Y}_{\mathcal{N}(i)} \qquad \mathbf{Y}_A \perp\!\!\!\perp \mathbf{Y}_B | \mathbf{Y}_C$$

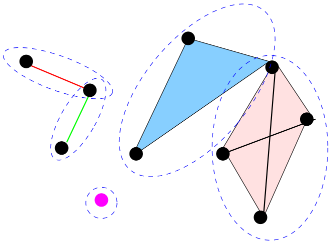
Graphical Models also known as Markov Random Fields.

Likelihood Function of Graphical Model

Hammersley-Clifford Theorem'71

Let P be conditional pdf of observations with graph $\mathcal{G}(\mathbf{v}_n)$, given locations $\mathbf{V}_n = \mathbf{v}_n$,

$$P(\mathbf{Y}_{\mathbf{v}_n}|\mathbf{V}_n = \mathbf{v}_n) = \frac{1}{Z}\exp[\sum_{c\in\mathcal{C}_n} \Psi_c(\mathbf{Y}_c)].$$



where $\mathcal{C}_n$ is the set of maximal cliques in $\mathcal{G}(\mathbf{v}_n)$ and Z is the normalization constant (partition function). Also, known as Gibbs distribution.

Graphical Models with Pairwise Interactions

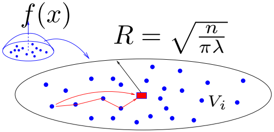
$$P(\mathbf{Y}_{\mathbf{v}_n}|\mathbf{V}_n = \mathbf{v}_n) = \frac{1}{Z}\exp[\sum_{(i,j)\in\mathcal{G}(\mathbf{v}_n)} \Psi_{i,j}(Y_i, Y_j)].$$

Example: Ising model has $\Psi_{i,j}(Y_i, Y_j) = \beta Y_i Y_j$ where β is called inverse temperature.

Euclidean Random Graphs

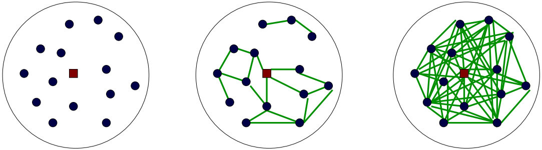
Random Node Placement

- Points $X_i \overset{\text{i.i.d.}}{\sim} f(x)$ on unit ball $\mathcal{B}_1$
- Fixed density λ: $V_i = \sqrt{\frac{n}{\lambda}} X_i$



Stabilizing graph (Penrose-Yukich)

- Local graph structure not affected by far away points (k-NNG, Disk)
- There is an a.s finite random radius R such that changes outside the ball do not affect the edges at origin.



M. D. Penrose and J. E. Yukich, "Weak Laws Of Large Numbers In Geometric Probability," *Annals of Applied probability*, vol. 13, no. 1, pp. 277-303, 2003

Illustration of Stabilization: 1-NNG

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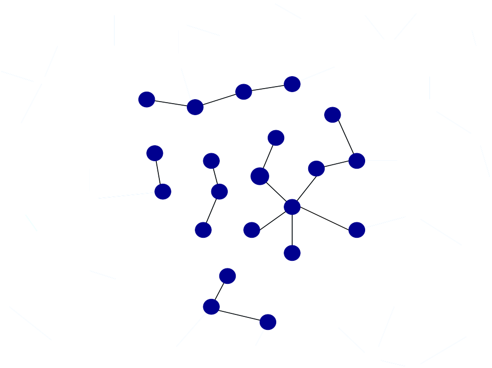

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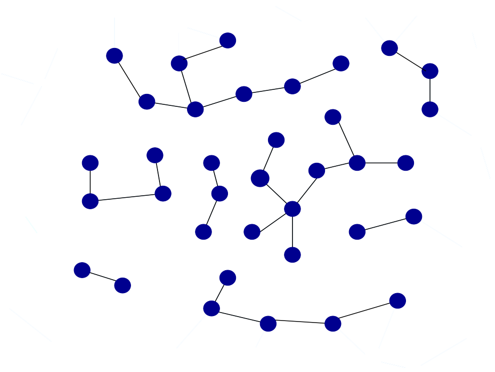

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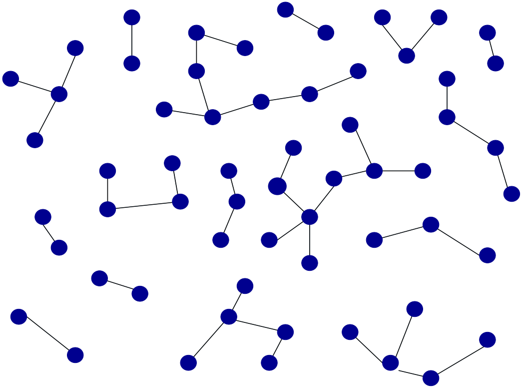

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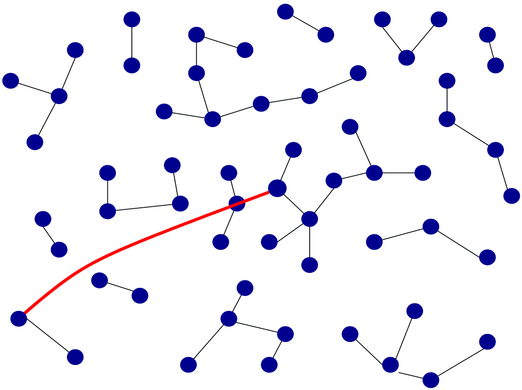

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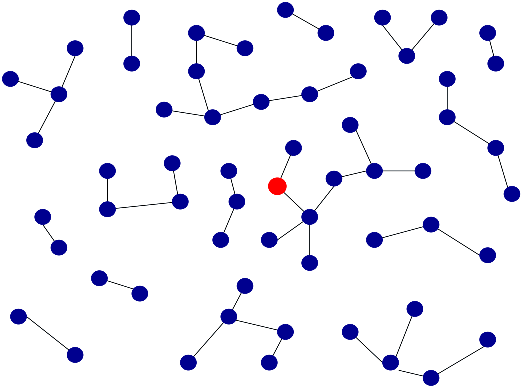

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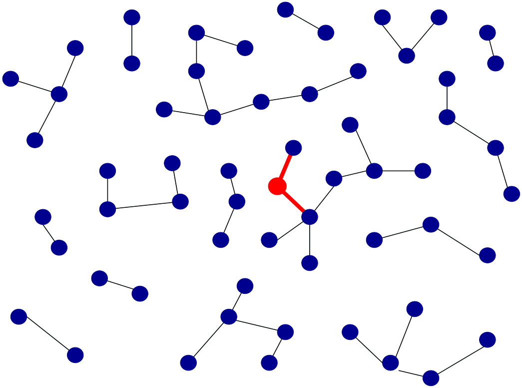

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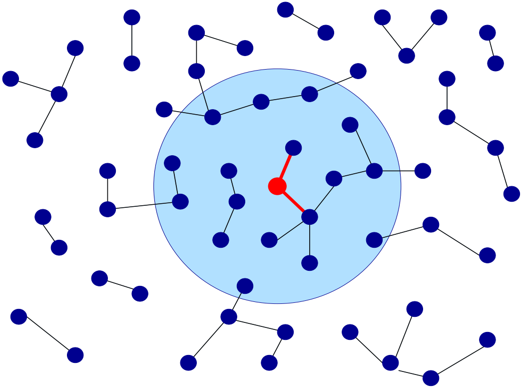

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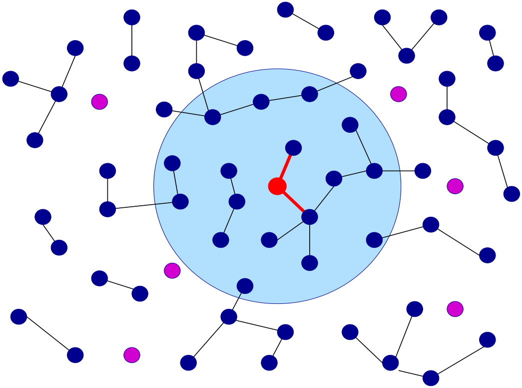

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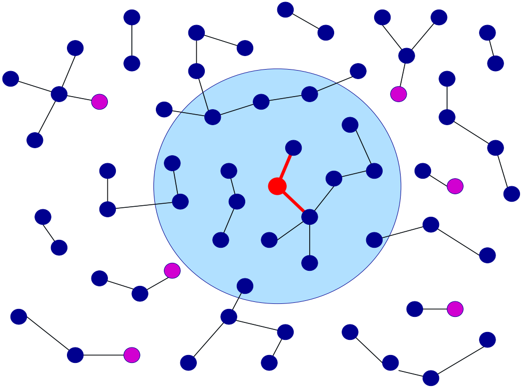

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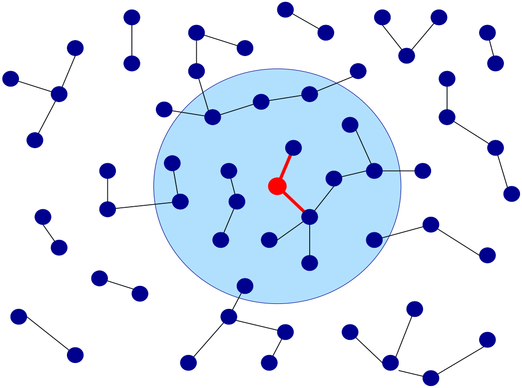

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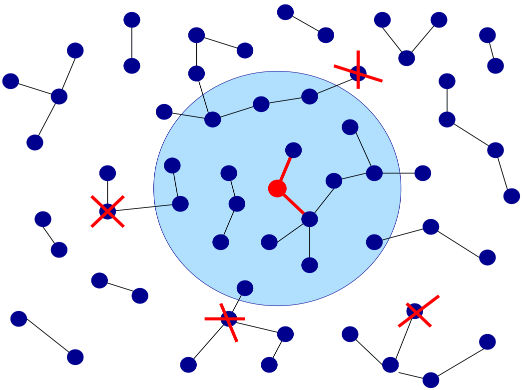

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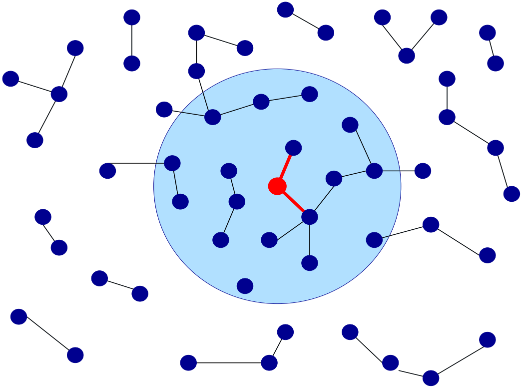

Outline

Functions on Random Spatial Graphical Models

- Locally-defined functions involving sums of terms at individual nodes

$$\xi((V_i, Y_{V_i}); (\mathbf{V}_n, \mathbf{Y}_{\mathbf{V}_n})) = \xi(Y_{V_i}; \mathbf{Y}_{\mathcal{N}(V_i)}).$$

  - Example: mean value of the observations.

Result

Under certain conditions

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \mathbb{E}[\xi((V_i, Y_{V_i}); (\mathbf{V}_n, \mathbf{Y}_{\mathbf{V}_n}))] = \xi_\infty.$$

$$\xi_\infty = \int_{B_1} \mathbb{E}[\xi((V_{\mathbf{0}}, Y_{\mathbf{0}}); (\mathcal{P}_{\lambda f(y)}, \mathbf{Y}_{\mathcal{P}_{\lambda f(y)}}))] f(y) dy,$$

where $\mathcal{P}_\lambda$ is Poisson process with intensity λ. For uniform node placement $(f \equiv 1)$,

$$\xi_\infty = \mathbb{E}[\xi((V_{\mathbf{0}}, Y_{\mathbf{0}}); (\mathcal{P}_\lambda, \mathbf{Y}_{\mathcal{P}_\lambda}))].$$

Limiting Constant ξ_∞ via Poissonization

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\mathbb{E}[\xi((V_i, Y_{V_i}); (\mathbf{V}_n, \mathbf{Y}_{\mathbf{V}_n}))] = \xi_\infty.$$

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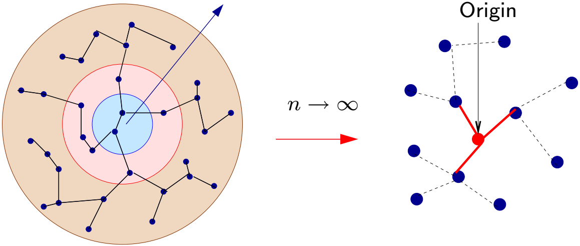
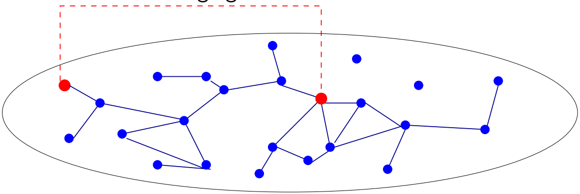


Conditions for Existence of Limits

- ξ has bounded moments.
- Graph $\mathcal{G}$ is Stabilizing, i.e., finite radius of stabilization.
- $P(\mathbf{Y}_{\mathbf{V}_n}|\mathbf{V}_n)$ is spatially mixing or in uniqueness regime.

Spatial Mixing or Uniqueness Regime



Asymptotic independence between observation at a node and observations at nodes far away

$$d_{\mathsf{TV}}(P[Y_v|\mathbf{Y}_V = \mathbf{y}_V], P[Y_v|\mathbf{Y}_V = \mathbf{z}_V]) \leq \delta(\mathsf{dist}_{\mathcal{G}}(v, V)),$$

for any two feasible configurations $\mathbf{y}_V, \mathbf{z}_V \in \mathcal{Y}^{|V|}$, such that $\lim_{s\to\infty} \delta(s) = 0$.

Outline

Maximum Degree in Random Graphs

Dobrushin Influence Coefficient

$$C_{i,j} := \max_{\substack{\mathbf{y},\mathbf{z}\in\mathcal{Y}^{|\mathbf{V}|-1} \\ y(k)=z(k),\,\forall k\neq j}} d_{\mathsf{TV}}(P[Y_i|\mathbf{Y}_{\mathbf{V}\setminus i}=\mathbf{y}], P[Y_i|\mathbf{Y}_{\mathbf{V}\setminus i}=\mathbf{z}]).$$

Dobrushin Condition for Spatial Mixing

$$\alpha := \max_{i\in\mathbf{V}_n} \sum_{j\in\mathbf{V}_n} C_{i,j}. \quad \alpha < 1.$$

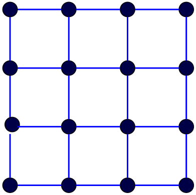
Maximum Degree of Dependency Graph

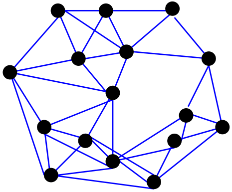
- Implies maximum degree of graph $\Delta < \infty$ as $n \to \infty$

  For Ising model, inverse temperature $\beta < \beta_c(\Delta)$

  k-NNG has $\Delta = (c+1)k$
- A graph with smaller Δ is spatially mixing for wider range of model parameters

Effects of Dependency Graph Randomness





Effect of Randomness in Dependency Graph

- Increases maximum degree in the graph
- More likely to have long range correlation and hence, limits do not exist

Graphs with Growing Maximum Degree

- Many random graphs have $\Delta \to \infty$ as $n \to \infty$.
  - Example: random geometric graph with unit threshold has $\Delta = \Theta(\frac{\log n}{\log \log n})$.
- Dobrushin's condition for spatial mixing requires $\Psi_{i,j} \to 0$ as $n \to \infty$.

Influence Coefficient

$$\rho(i) := \max_{\mathbf{y},\mathbf{z} \in \mathcal{Y}^{|\mathbf{V}|-1}} d_{\mathsf{TV}}(P[Y_i | \mathbf{Y}_{\mathbf{V} \setminus i} = \mathbf{y}], P[Y_i | \mathbf{Y}_{\mathbf{V} \setminus i} = \mathbf{z}]).$$

Degree Dependent Percolation

- Consider independent node percolation on $\mathcal{G}(\mathbf{V}_n)$ where probability of choosing a node i is $\rho(i)$
- If the resulting graph does not have a giant component, then corresponding graphical model is spatially mixing.

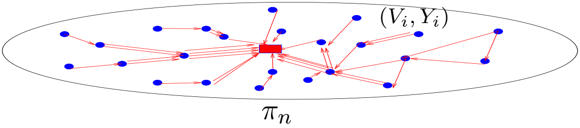
J. Van Den Berg and C. Maes, "Disagreement Percolation in the Study of Markov Fields," the Annals of Prob., vol. 22, no. 2, 1994.

Outline

Information Fusion for Inference of Graphical Models

Setup

- Binary hypothesis testing of two random spatial graphical models
- Fusion center makes the final decision from all the data
- Distributed computation to save energy costs, but require optimal detection at FC



Result

- When the graphical models have stabilizing graphs, constant scaling of average energy for optimal inference
- Efficient fusion policy with constant approximation ratio

A. Anandkumar, J.E. Yukich, L. Tong, A. Swami, "Energy scaling laws for distributed inference in random networks," *IEEE JSAC: Special Issues on Stochastic Geometry and Random Graphs*

Conclusion

Summary

- Modeled correlated observations as a graphical model with Euclidean random graphs
- Provided conditions for existence of locally-defined functions
- Conditions based on localization of graph and weak interactions (potentials)
- Related to degree-dependent percolation

Outlook

- Lower and upper bounds on functionals beyond uniqueness regime
- Behavior when graphs are not stabilizing.