Error Exponents for Composite Hypothesis Testing of Markov Forest Distributions

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ISIT (Jun 18, 2010)
Motivation

- Continuation of line of work on error exponents for learning tree-structured graphical models:

Instead of learning, we instead focus on hypothesis testing. Provides intuition for which classes of graphical models are easy for learning in terms of the detection error exponent. Is there a relation between the detection error exponent and the exponent associated to structure learning?
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- Is there a relation between the detection error exponent and the exponent associated to structure learning?
Graphical model: family of multivariate probability distributions that factorize according to a given graph $G = (V, E)$.

Vertices in the set $V = \{1, \ldots, d\}$ correspond to variables and edges in $E \subset \binom{V}{2}$ to conditional independences.
Background on Tree-Structured Graphical Models

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\[
P(x_1, x_2, x_3, x_4) = P_1(x_1) \times \frac{P_{1,2}(x_1, x_2)}{P_1(x_1)} \times \frac{P_{1,3}(x_1, x_3)}{P_1(x_1)} \times \frac{P_{1,4}(x_1, x_4)}{P_1(x_1)}.
\]
Learning vs Hypothesis Testing

- Canonical Problem: Given $x_1, \ldots, x_n \sim P$, learn structure of $P$. 

If $P$ is a tree, can use Chow and Liu (1968) as an efficient implementation of ML.

Denote set of distributions Markov on a tree $T_0 \in T$ as $D(T_0)$. Set of distributions Markov on any tree is $D(T)$.

Composite hypothesis testing problem considered here:

$H_0: x_1, \ldots, x_n \sim \Lambda_0 \subset D(T)$

$H_1: x_1, \ldots, x_n \sim \Lambda_1 \subset D(T)$

$\Lambda_i$ closed and $\Lambda_0 \cap \Lambda_1 = \emptyset$. 

4/17 Vincent Tan (SSG, LIDS, MIT)
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Definition of Worst-Case Type-II Error Exponent

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- Def: Type-II error exponent for a fixed \(Q \in \Lambda_1\) given \((A_n)\):

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J(\Lambda_0, Q; A_n) := \liminf_{n \to \infty} -\frac{1}{n} \log Q^n(A_n)
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- Def: Optimal Type-II error exponent

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J^*(\Lambda_0, Q) := \sup_{A_n} P_n(A_n) \leq \alpha, \forall P \in \Lambda_0
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- Def: Worst-Case Optimal Type-II error exponent

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J^*_w(\Lambda_0, \Lambda_1) := \inf_{Q \in \Lambda_1} J^*(\Lambda_0, Q)
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Optimizing distribution \(Q^*\) called the least favorable distribution.
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Why Difficult?

Many trees: If there are $d$ nodes, there are $d^d - 2$ trees!

Searching for the dominant error event may be intractable.

Natural Questions:
- Any closed-form expressions for the worst-case error exponent for special $\Lambda_0$, $\Lambda_1$?
- How does this depend on the true distribution?
- Connections to learning?
- Intuition and characterization of the least favorable distribution?
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A Simplification

Assume that $H_0$ is simple and $P$ is Markov on $T_0 = (V, E_0)$.

$$H_0 : x_1, \ldots, x_n \sim \{P\}$$

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\[ J^*(P) := J^*(\{P\}, \mathcal{D}(\mathcal{T}) \setminus \mathcal{D}(T_0)) \]
Setup for Main Result

For a non-edge $e' = (i, j)$, let $\text{Path}(e')$ be the unique path joining $i$ and $j$. Let $L(i, j)$ be the number of hops between $i$ and $j$. 
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![Diagram showing the setup with nodes $i$, $j$, $k$, and $l$, and edges $e' = (i, j)$, $e_1 = (i, k)$, and $e_2 = (k, j)$.]
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**Figure:** \( \text{Path}(e') = \{ (i, k), (k, j) \} \)
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\[
\begin{align*}
\text{Figure: } \text{Path}(e') &= \{(i,k), (k,j)\}
\end{align*}
\]

Mutual information of joint distribution \( P_e = P_{i,j} \) denoted as \( I(P_e) \).
Main Result

Proposition

\[ J^*(P) = \min_{e'=(i,j) \notin E_0} \min_{e \in \text{Path}(e')} \{ I(P_e) - I(P_{e'}) \}, \]

where \( L(i,j) = 2 \).
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Illustration:
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Illustration:

\[ J^*(P) = 4 - 3.9 = 0.1 \]
The least favorable distribution $Q^*$ is characterized by

$$E_{Q^*} = \arg\max_{E \neq E_0, E \text{ acyclic}} \sum_{e \in E} I(P_e)$$

a second-best max-weight spanning tree problem,
The least favorable distribution $Q^*$ is characterized by

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$$Q^*_i(x_i) = P_i(x_i), \quad \forall i \in V$$

$$Q^*_{i,j}(x_i, x_j) = P_{i,j}(x_i, x_j), \quad \forall (i, j) \in E_{Q^*}$$
**Proof Outline**

- Optimization for worst-case exponent is

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\inf_{Q \in \mathcal{D}(T) \setminus \mathcal{D}\left\{ T_0 \right\}} D(Q \| P)
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- Use tree decomposition (junction tree theorem)

\[ Q(x) = \prod_{i \in V(T)} Q_i(x_i) \prod_{(i,j) \in E(T)} \frac{Q_{i,j}(x_i, x_j)}{Q_i(x_i)Q_j(x_j)} \]
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- Data processing inequality.
Intuition

\[ J^*(P) = \min_{e'=(i,j) \notin E_0} \min_{e \in \text{Path}(e')} \{ I(P_e) - I(P_{e'}) \}, \]

- Smaller the difference between MI on true edge and MI on non-edge (along path), smaller the detection error exponent.
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- Detection error exponent depends only on bottleneck edges.
Comparison to Existing Results

\[ J^*(P) = \min_{e' = (i,j) \notin E_0} \min_{e \in \text{Path}(e')} \left\{ I(P_e) - I(P_{e'}) \right\}, \]

Intuitive in light of the Chow-Liu algorithm for learning trees.

\[ \hat{E}_{ML} := \arg\max_{E \neq E_0, E \text{ acyclic}} \sum_{e \in E} I(\hat{\mu}_e) \]

where \( \hat{\mu}_e \) is the pairwise type on edge \( e \).
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- Learning error exponent in very-noisy regime

\[ \tilde{K}(P) := \min_{e' \notin E_0} \min_{e \in \text{Path}(e')} \frac{(I(P_e) - I(P_{e'}))^2}{2 \text{Var}(S_e - S_{e'})} \]
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- \( J^*(P) \) and \( \tilde{K}(P) \) depend on the difference of mutual informations.
Performing the Hypothesis Test

- Known that the worst-case error exponent is achieved by the Hoeffding Test.
- But hard to implement for tree distributions.
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- The generalized likelihood ratio test (GLRT) has acceptance regions

\[ A_n := \left\{ x^n : \frac{1}{n} \log \frac{\max_{Q \in \Lambda_1} Q^n(x^n)}{\max_{P \in \Lambda_0} P^n(x^n)} \geq \gamma \right\} \]
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The Generalized Likelihood Ratio Test

- Denote the joint type of $x^n$ as $\hat{\mu} := \hat{\mu}(\cdot; x^n)$.
- Denote the pairwise type on $e$ as $\hat{\mu}_e$.
- True set of edges: $E_0$.

**Proposition**

*The GLRT simplifies as*

$$A_n = \left\{ x^n : \sum_{e \in E^*} I(\hat{\mu}_e) - \sum_{e \in E_0} I(\hat{\mu}_e) \geq \gamma \right\}$$

*where the “dominating edge set” is*

$$E^* = \arg\max_{E \neq E_0, \text{acyclic}} \sum_{e \in E} I(\hat{\mu}_e)$$
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Interpretation and Extensions

- **Easy to implement** the GLRT for testing between trees.

- Can find the tree structure $E^*$ **efficiently** once pairwise types $\hat{\mu}_e$ have been computed.

- Extensions to **forest-structured distributions** for error exponent and GLRT are straightforward.

- Recent work on high-dimensional **learning** of forest-structured distributions. \(^1\)

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\(^1\) VTan, A. Anandkumar, A. Willsky “Learning High-Dimensional Markov Forest Distributions: Analysis of Error Rates”, Submitted to JMLR, May 2010.
Concluding Remarks

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- Possible extension 2: Decomposable graphical models.