Error Exponents for Composite Hypothesis Testing of Markov Forest Distributions

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ISIT (Jun 18, 2010)

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- Continuation of line of work on error exponents for learning tree-structured graphical models:
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- Instead of learning, we instead focus on hypothesis testing.
- Provides intuition for which classes of graphical models are easy for learning in terms of the detection error exponent.
- Is there a relation between the detection error exponent and the exponent associated to structure learning?

Background on Tree-Structured Graphical Models

- Graphical model: family of multivariate probability distributions that factorize according to a given graph $G = (V, E)$.
- Vertices in the set $V = \{1, \ldots, d\}$ correspond to variables and edges in $E \subset \binom{V}{2}$ to conditional independences.

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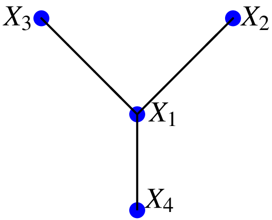

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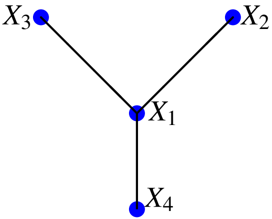

$$P(x_1, x_2, x_3, x_4) = P_1(x_1) \times \frac{P_{1,2}(x_1, x_2)}{P_1(x_1)} \times \frac{P_{1,3}(x_1, x_3)}{P_1(x_1)} \times \frac{P_{1,4}(x_1, x_4)}{P_1(x_1)}.$$

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- Composite hypothesis testing problem considered here:

$$H_0 : \mathbf{x}_1, \ldots, \mathbf{x}_n \sim \Lambda_0 \subset \mathcal{D}(\mathcal{T})$$
$$H_1 : \mathbf{x}_1, \ldots, \mathbf{x}_n \sim \Lambda_1 \subset \mathcal{D}(\mathcal{T})$$

- Λ_i closed and $\Lambda_0 \cap \Lambda_1 = \emptyset$.

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- Optimizing distribution Q^* called the least favorable distribution.

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- Any closed-form expressions for the worst-case error exponent for special Λ_0, Λ_1?
- How does this depend on the true distribution?
- Connections to learning?
- Intuition and characterization of the least favorable distribution?

A Simplification

Assume that H_0 is simple and P is Markov on $T_0 = (V, E_0)$.

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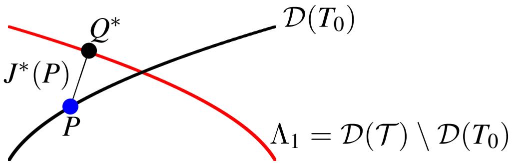

$J^*(P) := J^*(\{P\}, \mathcal{D}(\mathcal{T}) \setminus \mathcal{D}(T_0))$

Setup for Main Result

For a non-edge $e' = (i,j)$, let $\mathrm{Path}(e')$ be the unique path joining i and j.

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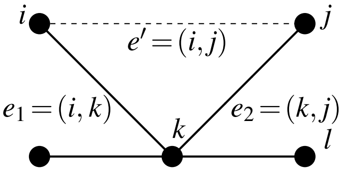

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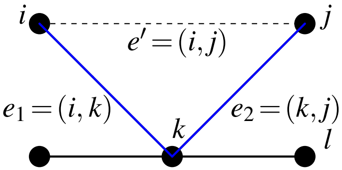

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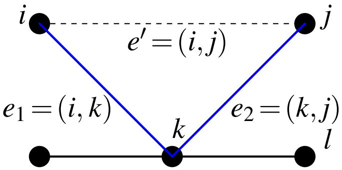

Figure: $\mathrm{Path}(e') = \{(i,k),(k,j)\}$

Mutual information of joint distribution $P_e = P_{i,j}$ denoted as $I(P_e)$.

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$$J^*(P) = \min_{\substack{e'=(i,j)\notin E_0 \\ L(i,j)=2}} \; \min_{e\in \mathrm{Path}(e')} \{I(P_e) - I(P_{e'})\},$$

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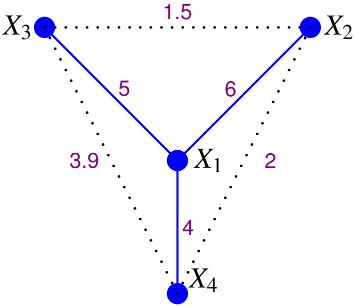

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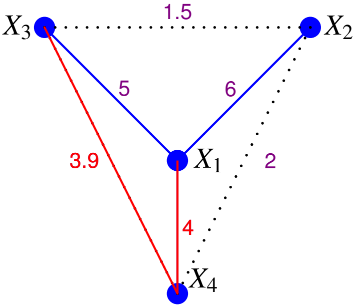

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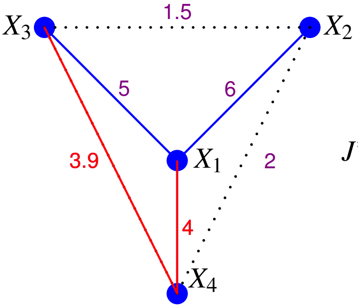

$$J^*(P) = 4 - 3.9 = 0.1$$

Least Favorable Distribution

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$$Q_i^*(x_i) = P_i(x_i), \qquad \forall\, i \in V$$
$$Q_{i,j}^*(x_i, x_j) = P_{i,j}(x_i, x_j), \qquad \forall\, (i,j) \in E_{Q^*}$$

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- Data processing inequality.

Intuition

$$J^*(P) = \min_{\substack{e'=(i,j)\notin E_0 \\ L(i,j)=2}} \min_{e \in \mathrm{Path}(e')} \{I(P_e) - I(P_{e'})\},$$

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- Detection error exponent depends only on bottleneck edges.

Comparison to Existing Results

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- Intuitive in light of the Chow-Liu algorithm for learning trees.

$$\hat{E}_{\mathrm{ML}} := \operatorname*{argmax}_{E\neq E_0,\, E \text{ acyclic}} \sum_{e\in E} I(\hat{\mu}_e)$$

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- $J^*(P)$ and $\widetilde{K}(P)$ depend on the difference of mutual informations.

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The Generalized Likelihood Ratio Test

- Denote the joint type of $\mathbf{x}^n$ as $\hat{\mu} := \hat{\mu}(\cdot\,;\,\mathbf{x}^n)$.
- Denote the pairwise type on e as $\hat{\mu}_e$.
- True set of edges: E_0.

Proposition

The GLRT simplifies as

$$\mathcal{A}_n = \left\{ \mathbf{x}^n : \sum_{e \in E^*} I(\hat{\mu}_e) - \sum_{e \in E_0} I(\hat{\mu}_e) \geq \gamma \right\}$$

where the "dominating edge set" is

$$E^* = \underset{E \neq E_0,\, E \text{ acyclic}}{\operatorname{argmax}} \sum_{e \in E} I(\hat{\mu}_e)$$

Interpretation and Extensions

- Easy to implement the GLRT for testing between trees.

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- Extensions to forest-structured distributions for error exponent and GLRT are straightforward.
- Recent work on high-dimensional learning of forest-structured distributions. [1]

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- Possible extension 2: Decomposable graphical models.